

Weights of the mod p kernel of the theta operators

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Abstract

We give some relations between the weights and the prime p of elements of the mod p kernel of the generalized theta operator $\Theta^{[j]}$. In order to construct examples of the mod p kernel of $\Theta^{[j]}$ from any modular form, we introduce new operators $A^{(j)}(M)$ and show the modularity of $F|A^{(j)}(M)$ when F is a modular form. Finally, we give some examples of the mod p kernel of $\Theta^{[j]}$ and the filtrations of some of them.

1 Introduction

Serre [26] developed the theory of p -adic and congruences for modular forms in one variable. In his paper, he showed some interesting properties of the θ -operator (Ramanujan's operator) acting on the q -expansions $\tilde{f} = \sum_{n \geq 0} \widetilde{a_f(n)} q^n \in \mathbb{F}_p[[q]]$ of modular forms mod p defined as

$$\theta = q \frac{d}{dq} : \tilde{f} \mapsto \theta(\tilde{f}) = \sum_{n \geq 1} \widetilde{na_f(n)} q^n.$$

As the one of his results, he showed that the weights (more precisely, the filtrations) of all elements of the mod p kernel of θ are divisible by p for the case of level 1. Moreover, Katz [16] showed that this property holds for the case of general level.

The first author and Nagaoka [7] extended the notion of the θ -operator to the case of Siegel modular forms of degree n . For this operator Θ (defined in [7]), several people found examples in the mod p kernel of Θ -operator (i.e., Siegel modular forms F satisfying $\Theta(F) \equiv 0 \pmod{p}$). The first author [3] observed that the Klingen-Eisenstein series of weight 12 arising from Ramanujan's Δ function is such an example for $p = 23$. After this, Mizumoto [21] found another example of weight 16 and $p = 31$, which comes from the Klingen-Eisenstein series arising from a cusp

form of weight 16. Recently, the authors, Kodama and Nagaoka [6, 17, 22, 25] constructed families of such examples of weight $\frac{n+p-1}{2}$ (resp. $\frac{n+3p-1}{2}$) and degree n if the weight is even (resp. odd).

A new feature in the Siegel case is that one should study also vector valued generalizations $\Theta^{[j]}$ of theta operators for $0 \leq j \leq n$; their p -adic properties were given in [9], e.g. $\Theta^{[1]}$ maps a Siegel modular form $\sum_T a_F(T)q^T$ to a formal series $\sum_T T a_F(T)q^T$ with coefficients in symmetric matrices of size n .

In this paper, we discuss the necessity (as in the one variable cases) of the relation between the weight and the prime p for an element of the mod p kernel of the generalized theta operators $\Theta^{[j]}$, in the case where the weight is small compared with p . We remark that Yamauchi [31] and Weissauer [29] also studied the necessity in the special cases $\Theta^{[1]}$ or Θ . Moreover we construct elements of the mod p kernel of $\Theta^{[j]}$ from arbitrary modular form. In order to do this, we introduce an operator $A^{(j)}(M)$ and study its properties (see Section 4). Finally, we give some examples of the mod p kernel of $\Theta^{[j]}$ and introduce the filtrations of some of them (Section 5, 6).

2 Preliminaries

2.1 Siegel modular forms

We denote by \mathbb{H}_n the Siegel upper half space of degree n . We define the action of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ on \mathbb{H}_n by $gZ = (AZ + B)(CZ + D)^{-1}$ for $Z \in \mathbb{H}_n$, $g \in \mathrm{Sp}_n(\mathbb{R})$. For a holomorphic function $F : \mathbb{H}_n \rightarrow \mathbb{C}$ and a matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$, we define the slash operator in the usual way;

$$F|_k g = j(g, Z)^{-k} F(gZ),$$

where $j(g, Z)$ is defined by $\det(CZ + D)$.

Let N be a natural number. In this paper, we deal with three types congruence subgroups of Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ as follows:

$$\begin{aligned} \Gamma^{(n)}(N) &:= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid B \equiv C \equiv 0_n \pmod{N}, A \equiv D \equiv 1_n \pmod{N} \right\}, \\ \Gamma_1^{(n)}(N) &:= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0_n \pmod{N}, A \equiv D \equiv 1_n \pmod{N} \right\}, \\ \Gamma_0^{(n)}(N) &:= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0_n \pmod{N} \right\}. \end{aligned}$$

Let Γ be the one of above modular groups of degree n with level N . For a natural number k and a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$, the space $M_k(\Gamma, \chi)$ of Siegel modular forms of weight k with character χ consists of all of holomorphic functions $F : \mathbb{H}_n \rightarrow \mathbb{C}$ satisfying

$$F|_k g = \chi(\det D) F(Z) \quad \text{for} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \quad (2.1)$$

If $n = 1$, the usual condition in the cusps should be added.

If $k = l/2$ is half-integral, then we assume that the level N of Γ satisfies $4 \mid N$. For $g \in \Gamma_0^{(n)}(4)$, we put

$$j_{1/2}(g, Z) := \theta^{(n)}(gZ)/\theta^{(n)}(Z),$$

where

$$\theta^{(n)}(Z) := \sum_{X \in \mathbb{Z}^n} e^{2\pi i t X Z X}.$$

Then it is known that

$$j_{1/2}(g, Z)^2 = \left(\frac{-4}{\det D} \right) \det(CZ + D) \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4),$$

where $\left(\frac{-4}{*} \right)$ is the Kronecker character for the discriminant -4 .

We define the slash operator for a holomorphic function $F : \mathbb{H}_n \rightarrow \mathbb{C}$ by

$$F|_k g := j_{1/2}(g, Z)^{-l} F(gZ) \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

We define $M_k(\Gamma, \chi)$ as the space of all of holomorphic functions $F : \mathbb{H}_n \rightarrow \mathbb{C}$ such that

$$F|_k g = \chi(\det D) F(Z) \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

For more details on Siegel modular forms of half-integral weight, see [1].

In both cases, when χ is a trivial character, we write simply $M_k(\Gamma)$ for $M_k(\Gamma, \chi)$. Any $F \in M_k(\Gamma, \chi)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \frac{1}{N}\Lambda_n} a_F(T) q^T, \quad q := e^{2\pi i \text{tr}(TZ)}, \quad Z \in \mathbb{H}_n,$$

where

$$\Lambda_n := \{T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}$$

(the lattice in $\text{Sym}_n(\mathbb{R})$ of half-integral, symmetric matrices). In particular, if Γ satisfies $\Gamma \supset \Gamma_1^{(n)}(N)$, then the Fourier expansion of F is given by the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_n} a_F(T) q^T.$$

We denote by Λ_n^+ the set of all positive definite elements of Λ_n . For a subring R of \mathbb{C} , let $M_k(\Gamma, \chi)_R \subset M_k(\Gamma, \chi)$ denote the R -module of all modular forms whose Fourier coefficients are in R .

2.2 Vector valued Siegel modular forms

For later use, we introduce the notion of vector valued Siegel modular forms briefly. Let $\Gamma \subset \Gamma_n$ be the one of subgroups of level N introduced in Subsection 2.1 and $\rho : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_{\mathbb{C}}(V_\rho)$ a polynomial representation of $\mathrm{GL}_n(\mathbb{C})$. A holomorphic function $F : \mathbb{H}_n \rightarrow V_\rho$ is said to be a vector valued Siegel modular form of automorphy factor ρ and of level Γ if and only if F satisfies the following property:

$$F(gZ) = \rho(CZ + D) F(Z) \quad \text{for all } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

If $n = 1$, we add the cusp condition.

As in the scalar valued case, a vector valued Siegel modular form F has the following Fourier expansion:

$$F(Z) = \sum_{0 \leq T \in \frac{1}{N}\Lambda_n} a_F(T) q^T, \quad Z \in \mathbb{H}_n, \quad a_F(T) \in V_\rho.$$

2.3 Congruences for modular forms

For a prime p we denote by ν_p the usual additive p -valuation of \mathbb{Q} , normalized by $\nu_p(p) = 1$. For a formal power series $F = \sum_{T \in \frac{1}{N}\Lambda_n} a_F(T) q^T$ with coefficients in \mathbb{Q} we define

$$\nu_p(F) := \inf \left\{ \nu_p(a_F(T)) \mid T \in \frac{1}{N}\Lambda_n \right\}.$$

If the power series is actually the Fourier expansion of a nonzero modular form, then $\nu_p(F)$ is always finite.

Let F_1, F_2 be two formal power series of the forms $F_i = \sum_{0 \leq T \in \frac{1}{N}\Lambda_n} a_{F_i}(T) q^T$ with $a_{F_i}(T) \in \mathbb{Q}$. We write

$$F_1 \equiv F_2 \pmod{p},$$

if and only if

$$\nu_p(F_1 - F_2) > \nu_p(F_1).$$

If $\nu_p(F_1) = 0$, this means $a_{F_1}(T) \equiv a_{F_2}(T) \pmod{p}$ for all $T \in \frac{1}{N}\Lambda_n$ with $T \geq 0$.

Let p be a prime and $\widetilde{M}_k(\Gamma)_{p^l}$ the space of modular forms mod p^l for Γ defined as

$$\widetilde{M}_k(\Gamma)_{p^l} := \{ \widetilde{F} \mid F \in M_k(\Gamma)_{\mathbb{Z}_{(p)}} \},$$

where $\widetilde{F} := \sum_T \widetilde{a_F(T)} q^T$ and $\widetilde{a_F(T)} := a_F(T) \pmod{p^l}$. We put

$$\widetilde{M}(\Gamma)_{p^l} := \sum_{k \in \mathbb{Z}_{\geq 0}} \widetilde{M}_k(\Gamma)_{p^l}.$$

Let ω_N be the filtration of modular forms mod p introduced by Serre, Swinnerton-Dyer: For a formal power series of the form $F = \sum_{T \in \Lambda_n} a_F(T)q^T$ (not constant modulo p) with $a_F(T) \in \mathbb{Z}_{(p)}$, we define

$$\omega_N(F) := \inf\{k \in \mathbb{Z}_{\geq 1} \mid \tilde{F} \in \widetilde{M}_k(\Gamma_1^{(n)}(N))_p\}.$$

If $F \equiv c \pmod{p}$ for some $c \in \mathbb{Z}_{(p)}$, then we regard as $\omega_N(F) = 0$.

It follows from [8, 15] immediately that

Proposition 2.1. Let p be an odd prime, N a positive integer with $p \nmid N$ and $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Then

$$\omega_N(F) \equiv k \pmod{p-1}.$$

In particular, if $k < p$ and $F \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$, then $k = \omega_N(F)$.

Definition 2.2. A formal power series of the form $F = \sum_{T \in \Lambda_n} a_F(T)q^T$ with $a_F(T) \in \mathbb{Z}_p$ is called a *p-adic modular form (of degree n)* if there exists a sequence $\{G_l \in M_{k_l}(\Gamma_n)_{\mathbb{Z}_{(p)}}\}$ of modular forms such that

$$\lim_{l \rightarrow \infty} G_l = F \quad (p\text{-adically}),$$

in other words,

$$\nu_p(F - G_l) \rightarrow \infty \quad (l \rightarrow \infty).$$

Theorem 2.3 ([9]). Let p be a prime with $p \geq n+3$. Then any $F \in M_k(\Gamma_0^{(n)}(p^m))_{\mathbb{Z}_{(p)}}$ ($m \geq 0$) is a *p-adic modular form*. In particular, we have $\widetilde{M}(\Gamma_0^{(n)}(p^m))_{p^l} \subset \widetilde{M}(\Gamma_n)_{p^l}$ for any $l \geq 0$ and $m \geq 1$.

2.4 Theta operators and their properties

To define the operators $\Theta^{[j]}$ we need some notation: For a symmetric matrix T of size n we denote by $T^{[j]}$ the matrix of size $\binom{n}{j} \times \binom{n}{j}$ whose entries are given by the determinants of all submatrices of size j . Then we can explain $\Theta^{[j]}$ by

$$F = \sum_T a_F(T)q^T \longmapsto \Theta^{[j]}(F) := \sum_T T^{[j]} \cdot a_F(T)q^T$$

for any formal power series of the type above.

These operators were introduced in [9], and it was shown, that they define (vector valued) modular forms mod p , when applied to modular forms. Note that $\Theta^{[n]}$ is the Θ -operator defined in [7], i.e., for a formal Fourier series $F = \sum_{T \in \Lambda_n} a_F(T)q^T$, we have

$$\Theta^{[n]}(F) = \sum_{T \in \Lambda_n} (\det T) a_F(T)q^T.$$

Hence, we write simply as Θ for $\Theta^{[n]}$.

For $0 \leq j \leq n$, we observe the following (obvious) properties of these operators with respect to congruences:

Proposition 2.4. (1) $\Theta^{[j]}(F) \not\equiv 0 \pmod{p}$ is equivalent to the existence of $T \in \Lambda_n$ and a $j \times j$ submatrix R of T such that $a_F(T) \not\equiv 0 \pmod{p}$ and $p \nmid \det R$.

(2) $\Theta^{[j]}(F) \equiv 0 \pmod{p}$ implies $\Theta^{[j+1]}(F) \equiv 0 \pmod{p}$.

(3) $\Theta^{[j]}(F)$ is a mod p singular if and only if $T^{[j]}a_F(T) \equiv 0 \pmod{p}$ for all $T \in \Lambda_n^+$.

(4) Let $j \geq 0$ be an integer. If $\Theta^{[j]}(F)$ is mod p singular with p -rank r_p , then $\Theta^{[r_p+1]}(F) \equiv 0 \pmod{p}$.

The following theorem is due to Katz:

Theorem 2.5 (Katz [16] (cf. Serre [26])). Let p be an arbitrary prime and N a positive integer such that $N \geq 3$ and $p \nmid N$. For $f \in M_k(\Gamma_1^{(1)}(N))_{\mathbb{Z}_{(p)}}$ ($k \in \mathbb{Z}_{\geq 1}$), suppose that $\Theta^{[1]}(f) \equiv 0 \pmod{p}$. Then we have $p \mid \omega_N(f)$.

Remark 2.6. In this case (the degree is 1), the operator $\Theta^{[1]}$ is the usual Ramanujan's operator θ .

Theorem 2.7 ([7]). Let p be a prime with $p \geq n + 3$. If $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, then $\widetilde{\Theta}(F) \in \widetilde{M}_{k+p+1}(\Gamma_n)_p$. In particular $\omega_1(\Theta(F)) \leq k + p + 1$ holds. Therefore we can regard as

$$\Theta : \widetilde{M}(\Gamma_n)_p \longrightarrow \widetilde{M}(\Gamma_n)_p.$$

We introduce a relation between mod p singular modular forms and the mod p kernel of $\Theta^{[j]}$:

Proposition 2.8. Let p be an odd prime and N a positive integer with $p \nmid N$. For a positive integer k , assume that $F \in M_k(\Gamma^{(n)}(N))_{\mathbb{Z}_{(p)}}$ is mod p singular of p -rank r_p with $k \not\equiv \frac{r_p}{2} \pmod{2}$. Then we have $\Theta^{[r_p]}(\Phi^{(n-r_p)}(F)) \equiv 0 \pmod{p}$. Here Φ is the Siegel Φ -operator and hence $\Phi^{(n-r_p)}(F) \in M_k(\Gamma^{(r_p)}(N))$.

Remark 2.9. If $F \in M_k(\Gamma^{(n)}(N))$ is mod p singular of p -rank r_p , then we have $2k - r_p \equiv 0 \pmod{p-1}$ by the result of [5]. Therefore in this case r_p should automatically be even.

Proof. By taking Φ -operator several times, we may suppose that the p -rank r_p is $n - 1$. Then it suffices to prove that $\Theta^{[n-1]}(\Phi(F)) \equiv 0 \pmod{p}$. Moreover, by taking $F(NZ) \in M_k(\Gamma_1^{(n)}(N^2))$ when F is of $\Gamma^{(n)}(N)$, it suffices to prove it for $F \in M_k(\Gamma_1^{(n)}(N))$.

For any $T_0 \in \Lambda_n$ with $\text{rank}(T_0) = n - 1$ satisfying $a_F(T_0) \not\equiv 0 \pmod{p}$, we may assume that $T_0 = \begin{pmatrix} 0 & 0 \\ 0 & M_0 \end{pmatrix}$ for some $M_0 \in \Lambda_{n-1}^+$. We shall prove $p \mid \det M_0$ for any such M_0 .

Recall that F has a Fourier-Jacobi expansion of the form

$$F(Z) = \sum_{M \in \Lambda_{n-1}} \varphi_T(\tau, z) e^{2\pi i \text{tr}(M \cdot \tau')}.$$

Here, we decomposed $\mathbb{H}_n \ni Z = \begin{pmatrix} \tau & t\mathfrak{z} \\ \mathfrak{z} & \tau' \end{pmatrix}$ for $\tau \in \mathbb{H}_1$, $\tau' \in \mathbb{H}_{n-1}$. Pick up M_0 -th Fourier-Jacobi coefficient and consider its theta expansion;

$$\varphi_{M_0}(\tau, \mathfrak{z}) = \sum_{\mu} h_{\mu}(\tau) \Theta_{M_0}[\mu](\tau, \mathfrak{z})$$

where μ runs over all elements of $\mathbb{Z}^{(1, n-1)} \cdot (2M_0) \setminus \mathbb{Z}^{(1, n-1)}$ and

$$h_{\mu}(\tau) = \sum_{l=0}^{\infty} a_F \left(\begin{pmatrix} l & \frac{\mu}{2} \\ \frac{t\mu}{2} & M_0 \end{pmatrix} \right) e^{2\pi i (l - \frac{1}{4} M_0^{-1} [t\mu]) \tau}.$$

From the mod p singularity of F , the above h_0 satisfies that

$$h_0 \equiv c \not\equiv 0 \pmod{p}$$

Moreover h_0 is a modular form of weight $k - \frac{n-1}{2} \in \mathbb{Z}_{\geq 1}$ for $\Gamma_0^{(1)}(NL)$ by Lemma 5.1 in [5]. Here L is the level of M_0 . If $p \nmid L$ then we have $k - \frac{n-1}{2} \equiv 0 \pmod{p-1}$ by Katz [16]. However this is impossible because of $k \not\equiv \frac{r_p}{2} \pmod{2}$. Hence $p \mid L$ follows. In particular we have $p \mid \det M_0$. This completes the proof of Proposition 2.8. \square

3 Main results and their proofs

3.1 Main results

For any $T \in \Lambda_n$, we denote by $\varepsilon(T)$ the content of T defined as

$$\varepsilon(T) := \max\{d \in \mathbb{Z}_{\geq 1} \mid d^{-1}T \in \Lambda_n\}.$$

Let F be a scalar valued modular form. If $F \not\equiv 0 \pmod{p}$ and $\Theta(F) \equiv 0 \pmod{p}$, then there are three possibilities as follows;

- (a) For any $T \in \Lambda_n^+$ we have $a_F(T) \equiv 0 \pmod{p}$.
- (b) For any $T \in \Lambda_n^+$ with $a_F(T) \not\equiv 0 \pmod{p}$, we have $p \mid \varepsilon(T)$.
- (c) There exists $T \in \Lambda_n^+$ such that $a_F(T) \not\equiv 0 \pmod{p}$ and $p \nmid \varepsilon(T)$.

A modular form F of the type (a) is called “mod p singular”. In this case, the authors discussed the possible weight in [5]. Therefore, the main purpose of this paper is to consider the types (b) and (c).

The first main result concerns F of the type (b), but under a condition on k . Note that the condition (b) is equivalent to $\Theta^{[1]}(F)$ is mod p singular, i.e. the vector valued modular form $\Theta^{[1]}(F) \pmod{p}$ satisfies the condition (a) above, because of Proposition 2.4 (3).

Theorem 3.1. Let p be a prime with $p \geq 3$ and N a positive integer satisfying that $N \geq 3$ and $p \nmid N$ or $N = 1$. For a positive integer k , let $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$. Assume that $F \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$.

(1) If $\Theta^{[1]}(F) \equiv 0 \pmod{p}$ and

$$\begin{cases} 0 < k < 2p - 1 & (k \text{ odd}), \\ 0 < k < 3p - 1 & (k \text{ even}) \end{cases} \text{ then } k = \begin{cases} p & (k \text{ odd}), \\ 2p & (k \text{ even}). \end{cases}$$

(2) If $\Theta^{[1]}(F) \equiv 0 \pmod{p}$, $0 < k < p^2 - p + 1$ and $p \mid k$ then $k = \omega_N(F)$.

(3) If $\Theta^{[1]}(F)$ is non-trivial mod p singular of p -rank r_p , then $2k - r_p \equiv 0 \pmod{p - 1}$.

Remark 3.2. (1) A modular form F satisfying $\Theta^{[1]}(F) \equiv 0 \pmod{p}$ is called “totally p -singular” by Weissauer [29]. Weissauer [29] also obtained the similar statements (at least for the case of level 1) under a certain condition on the largeness of p , in geometrical terminology. Our statement is phrased in classical (elementary) language. (2) There exists a mod p singular modular form $F (\not\equiv 0 \pmod{p})$ such that $\Theta^{[1]}(F) \equiv 0 \pmod{p}$. In fact, we can construct such an example in the following way: For any mod p singular modular form $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, we consider

$$G := \sum_{T \in \Lambda_n} a_F(pT) q^{pT} \in M_k(\Gamma_0^{(n)}(p^2)).$$

Applying Theorem 2.3, we can take $H \in M_{k'}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ such that $H \equiv G \pmod{p}$. Then H is a mod p singular modular form such that $\Theta^{[1]}(H) \equiv 0 \pmod{p}$. For the existence of mod p singular modular forms and for their possible weights, see [5].

The statement (3) in this theorem follows immediately from a property on mod p singular vector valued Siegel modular forms, which is a generalization of the result in [5]:

Theorem 3.3. Let p be an odd prime and k a positive integer. Let N be a positive integer with $p \nmid N$ and $F \in M_k(\Gamma_1^{(n)}(N))$. Suppose that $\Theta^{[j]}(F)$ is mod p singular of p -rank r_p . Then

$$2k \equiv r_p \pmod{p - 1}$$

holds.

Remark 3.4. We may allow the modular group to be of type $\Gamma_1^{(n)}(N) \cap \Gamma_0^{(n)}(p^l)$ with N coprime to p . We may also allow quadratic nebentypus modulo p .

The second main result concerns F of the type (c). We remark that $p \nmid \varepsilon(T)$ for $T \in \Lambda_n$ is equivalent to the existence of j with $1 \leq j \leq n - 1$ such that $p \nmid T^{[j]}$, where we write $p \mid T^{[j]}$ if p divides all entries of $T^{[j]}$, otherwise we write $p \nmid T^{[j]}$.

Moreover the existence of j and $T \in \Lambda_n^+$ with $p \nmid T^{[j]}$ such that $a_F(T) \not\equiv 0 \pmod p$, implies $\Theta^{[j]}(F) \not\equiv 0 \pmod p$. Namely we have

$$\begin{aligned} & \exists T \in \Lambda_n^+ \text{ s.t. } a_F(T) \not\equiv 0 \pmod p, \ p \nmid \varepsilon(T) \\ \iff & \exists j \ (1 \leq j \leq n-1), \ \exists T \in \Lambda_n^+ \text{ s.t. } a_F(T) \not\equiv 0 \pmod p, \ p \nmid T^{[j]} \\ \implies & \exists j \ (1 \leq j \leq n-1) \text{ s.t. } \Theta^{[j]}(F) \not\equiv 0 \pmod p. \end{aligned}$$

Note also that the converse of the last right arrow is not assured in general.

For any F of the type (c), we can find j such that $\Theta^{[j]}(F) \not\equiv 0 \pmod p$ and $\Theta^{[j+1]}(F) \equiv 0 \pmod p$. Then we have the following statement:

Theorem 3.5. Let p be a prime with $p \geq 3$ and N a positive integer with $p \nmid N$. Let n, j and k be positive integers such that $j < n$. Assume that $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$ satisfies $\Theta^{[j+1]}(F) \equiv 0 \pmod p$ and $\Theta^{[j]}(F) \not\equiv 0 \pmod p$.

(1) If

$$\begin{cases} k < p + (j-1)/2 & (j \text{ odd}), \\ k < 2p + (j-2)/2 & (j \text{ even, } k - j/2 \text{ odd}), \\ k < 3p + (j-2)/2 & (j \text{ even, } k - j/2 \text{ even}) \end{cases} \quad \text{then}$$

$$\begin{cases} 2k - j = p & (j \text{ odd}), \\ k - j/2 = p & (j \text{ even, } k - j/2 \text{ odd}), \\ k - j/2 \equiv 0 \pmod{p-1} \text{ or } k - j/2 = 2p & (j \text{ even, } k - j/2 \text{ even}). \end{cases}$$

(2) If

$$\begin{cases} 2k - j < p^2 - p + 1 & (j \text{ odd}), \\ k - j/2 < p^2 - p & (j \text{ even}) \end{cases} \quad \text{and} \quad p \mid (2k - j) \quad \text{then} \quad k = \omega_N(F).$$

In a more general situation, we predict that

Conjecture 3.6. Let p be a prime and n, j and k be positive integers with $j < n$. Let “ k be sufficiently small compared with p ”. Assume that $F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}}$ ($k \in \mathbb{Z}_{\geq 1}$) satisfies $\Theta^{[j+1]}(F) \equiv 0 \pmod p$ and $\Theta^{[j]}(F) \not\equiv 0 \pmod p$. Then we have

$$p \mid (2\omega_N(F) - j)$$

Therefore, we can regard Theorem 3.5 as an example which supports this conjecture.

Remark 3.7. (1) If the weight is large compared with p , the statement of this conjecture is not true. We will show this for the case of degree 2 and level 1, by numerical examples in Subsections 5.3 and 5.4.

(2) Yamauchi [31] concluded the similar statements as in the two theorems above for the case of degree 2, without condition on the smallness of k compared with p , but under a certain geometrical non-vanishing condition. The proof is also algebraic geometrical.

We summarize the simplest case $n = 2$ in Theorems 3.1 and 3.5 as follows:

Corollary 3.8. Let p be a prime with $p \geq 3$ and N a positive integer with $p \nmid N$. For a positive integer k , assume that $F \in M_k(\Gamma_1^{(2)}(N))_{\mathbb{Z}_{(p)}}$ satisfies $\Theta(F) \equiv 0 \pmod{p}$.

(1) If $k < p$ and there exists $T \in \Lambda_2^+$ with $p \nmid \varepsilon(T)$ satisfying $a_F(T) \not\equiv 0 \pmod{p}$, then we have $2k - 1 = p$.

(2) If we have $p \mid \varepsilon(T)$ for any $T \in \Lambda_2^+$ with $a_F(T) \not\equiv 0 \pmod{p}$, then $\Theta^{[1]}(F) \equiv 0 \pmod{p}$. In particular, if

$$\begin{cases} 0 < k < 2p - 1 & (k \text{ odd}), \\ 0 < k < 3p - 1 & (k \text{ even}) \end{cases} \text{ then } k = \begin{cases} p & (k \text{ odd}), \\ 2p & (k \text{ even}). \end{cases}$$

3.2 Proof of Theorem 3.3

Since Theorem 3.1 (3) follows from Theorem 3.3, we start with proving Theorem 3.3.

We observe that the Fourier expansion of $\Theta^{[j]}(F)$ runs only over elements of Λ_n with $\text{rank}(T) \geq j$, therefore only the case $j \leq r_p$ is of interest for us. Also it may be convenient to reduce the claim to the case $r_p = n - 1$ by applying the Siegel Φ -operator several times; for details on the Siegel Φ -operator in the vector-valued case we refer to [14, 30]. We just mention that for $j < n$ we may identify

$$\Phi \left(\sum_{T \in \Lambda_n} T^{[j]} \cdot a_F(T) q^T \right)$$

with

$$\sum_{S \in \Lambda_{n-1}} S^{[j]} \cdot a_F \left(\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \right) q^S.$$

We introduce some useful notation following [14]: For a n -rowed matrix M and two subsets P, Q of $\{1, \dots, n\}$ with t elements we denote by $M_{(P,Q)}^{[t]}$ the determinant of the t -rowed matrix M_Q^P which we obtain from M by deleting all rows which do not belong to P and all columns which do not belong to Q .

Now start from the Fourier expansion $F = \sum a_F(T) q^T$; there exists $T_0 \in \Lambda_n$ with $\text{rank } n - 1$ such that $T_0^{[j]} \cdot a_F(T_0)$ is not congruent zero modulo p . We may assume that T_0 is of the form

$$T_0 = \begin{pmatrix} 0 & 0 \\ 0 & M_0 \end{pmatrix} \quad (M_0 \in \Lambda_{n-1}^+).$$

The property of T_0 from above implies that there is at least one entry of the matrix $T_0^{[j]}$ which is not congruent zero modulo p , i.e. there exist subsets a^o, b^o of $\{1, \dots, n\}$ with $\det(T_{0b^o}^{a^o}) \not\equiv 0 \pmod{p}$. From the special shape of T_0 it follows that both a^o and b^o

are subsets of $\{2, \dots, n\}$, i.e. the (a^o, b^o) -entry of $T_0^{[j]}$ is a determinant of a submatrix of M_0 , which we call d_0 .

We decompose $Z \in \mathbb{H}_n$ as $Z = \begin{pmatrix} \tau & \mathfrak{z} \\ \mathfrak{z}^t & \tau' \end{pmatrix}$ with $\tau' \in \mathbb{H}_{n-1}$ and study the Fourier Jacobi coefficient

$$\varphi_{M_0}(\tau, \mathfrak{z}) e^{2\pi i \text{tr}(M_0 \tau')}$$

viewed as a subseries of the Fourier expansion of F . We apply the operator $\Theta^{[j]}$ to this subseries of F ; then its (a^o, b^o) entry is just

$$d_0 \cdot \varphi_{M_0}(\tau, \mathfrak{z}) e^{2\pi i \text{tr}(M_0 \tau')}$$

Now we may proceed as in [5]: For all $R \in \Lambda_n$ the R -th Fourier coefficients of this series should be congruent zero modulo p unless R is of rank $n-1$. This implies that in the theta expansion of φ_{M_0} a modular form h_0 of weight $k - \frac{n-1}{2}$ appears, which should be congruent to a (nonzero) constant modulo p . The requested congruence follows from this as in [5]. \square

3.3 Proof of Theorem 3.1

(1) From Proposition 2.4, the assumption $\Theta^{[1]}(F) \equiv 0 \pmod{p}$ implies that, for any $T \in \Lambda_n$ satisfying $a_F(T) \not\equiv 0 \pmod{p}$, we have $p \mid \varepsilon(T)$. In particular, all diagonal components of such T are divisible by p . We fix one of them such that $T \neq 0_n$ and denote it by T_0 . The existence of such T_0 is guaranteed by the assumption $F \not\equiv c \pmod{p}$. Let $\text{diag}(T_0) = (d_1, d_2, \dots, d_n)$ (with $p \mid d_i$ for any i). Since $T_0 \neq 0_n$, we may assume that $d_1 > 0$ by changing T_0 in its $\text{GL}_n(\mathbb{Z})$ -equivalence class.

Consider the integral extract for T_0 defined by the first author and Nagaoka [8];

$$\begin{aligned} f(\tau) &:= \sum_l a(l) q^l, \\ a(l) &:= \sum_T a_F(T), \end{aligned}$$

where T runs over all positive semi-definite elements of Λ_n satisfying

$$T \equiv T_0 \pmod{M}, \quad \text{diag}(T) = (l, d_2, \dots, d_n)$$

and M is a large enough such that $(p \det T_0, M) = 1$ and $a(d_1) \equiv a_f(T_0) \not\equiv 0 \pmod{p}$. Note that $f \in M_k(\Gamma_1^{(1)}(NM^2))$. Then $p \mid l$ when $a(l) \not\equiv 0 \pmod{p}$ and $f \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$. Hence we have $\Theta^{[1]}(f) \equiv 0 \pmod{p}$ and $\omega_{NM^2}(f) > 0$. Applying Theorem 2.5, we obtain $p \mid \omega_{NM^2}(f)$. By Proposition 2.1, $\omega_{NM^2}(f) \equiv k \pmod{p-1}$ and therefore $\omega_{NM^2}(f)$ and k have the same parity.

If k is odd, then $0 < \omega_{NM^2}(f) \leq k < 2p-1$ and $p \mid \omega_{NM^2}(f)$. In this case we have $\omega_{NM^2}(f) = p$. Therefore $k = p$ or $k = 2p-1$. By the assumption $k < 2p-1$, we obtain $k = p$.

If k is even, then $\omega_{NM^2}(f)$ is even such that $0 < \omega_{NM^2}(f) \leq k < 3p - 1$ and $p \mid \omega_{NM^2}(f)$. In this case we have $\omega_{NM^2}(f) = 2p$. Therefore $k = 2p$ or $k = 3p - 1$. Since the assumption $k < 3p - 1$, we obtain $k = 2p$. This completes the proof of (1) in Theorem 3.1.

(2) As in the proof of (1), we can take $f \in M_k(\Gamma_1^{(1)}(NM^2))$ ($p \nmid M$) such that $\Theta^{[1]}(f) \equiv 0 \pmod{p}$, $f \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$ and

$$0 < \omega_{NM^2}(f) \leq \omega_N(F) \leq k.$$

To prove $k = \omega_N(F)$, we may prove $k = \omega_{NM^2}(f)$. Now we obtain $p \mid \omega_{NM^2}(f)$ by Theorem 2.5. By the assumption, we have $\omega_{NM^2}(f) \equiv k \pmod{p(p-1)}$, because $\omega_{NM^2}(f) \equiv k \equiv 0 \pmod{p}$ and $\omega_{NM^2}(f) \equiv k \pmod{p-1}$. Then there exists $t \geq 0$ such that $k = \omega_{NM^2}(f) + tp(p-1)$. However, from $0 < k < p^2 - p + 1$, we have $t = 0$ and hence $k = \omega_{NM^2}(f)$. This completes the proof of (2) in Theorem 3.1.

(3) The statement follows immediately from Theorem 3.3. \square

3.4 Proof of Theorem 3.5

(1) By Proposition 2.4 and the assumption $\Theta^{[j]}(F) \not\equiv 0 \pmod{p}$, there exist $T \in \Lambda_n$ and a $j \times j$ submatrix M of T such that $a_F(T) \not\equiv 0 \pmod{p}$ and $p \nmid \det M$.

By changing T in its $\text{GL}_n(\mathbb{Z})$ -equivalence class, we may assume that there is a principal submatrix M of size j in T such that $p \nmid \det M$, where principal matrix is defined as a matrix obtained by omitting the same columns and rows. In particular, T is equivalent over $\text{GL}_n(\mathbb{Z})$ to a matrix of the form

$$T_0 = \begin{pmatrix} 0 & 0 \\ 0 & M_0 \end{pmatrix}$$

with $M_0 \in \Lambda_j$, $p \nmid \det M_0$ and $a_F(T_0) \not\equiv 0 \pmod{p}$.

The Fourier-Jacobi expansion of F can be written in the form

$$F(Z) = \sum_{M \in \Lambda_j} \varphi_M(\tau, \mathfrak{z}) e^{2\pi i \text{tr}(M \cdot \tau')}.$$

Here, we decomposed $\mathbb{H}_n \ni Z = \begin{pmatrix} \tau & t\mathfrak{z} \\ \mathfrak{z} & \tau' \end{pmatrix}$ for $\tau \in \mathbb{H}_{n-j}$ and $\tau' \in \mathbb{H}_j$.

We consider the M_0 -th Fourier-Jacobi coefficient

$$\varphi_{M_0}(\tau, \mathfrak{z}) = \sum_{\mu} h_{\mu}(\tau) \Theta_{M_0}[\mu](\tau, \mathfrak{z}),$$

where μ runs over all elements of $\mathbb{Z}^{(n-j, j)} \cdot (2M_0) \setminus \mathbb{Z}^{(n-j, j)}$ and

$$h_{\mu}(\tau) = \sum_{L \in \Lambda_{n-j}} a_F \left(\begin{pmatrix} L & \frac{\mu}{2} \\ t\frac{\mu}{2} & M_0 \end{pmatrix} \right) e^{2\pi i \text{tr}((L - \frac{1}{4}M_0^{-1}[t\mu])\tau)}.$$

Then h_μ is a modular form of weight $k - \frac{j}{2}$ for $\Gamma^{(n-j)}(4NQ)$, where Q is the level of M_0 and we have $p \nmid Q$. Hence we have

$$H_\mu := h_\mu(4NQ\tau) \in M_{k-\frac{j}{2}}(\Gamma_1^{(n-j)}(4^2N^2Q^2))_{\mathbb{Z}_{(p)}}.$$

Now we prove

Lemma 3.9. If the Fourier coefficient of H_μ at $L - \frac{1}{4}M_0^{-1}[t\mu]$ is nonzero modulo p , then all entries of

$$L - \frac{1}{4}M_0^{-1}[t\mu]$$

are divisible by p . In particular, by Proposition 2.4, we have

$$\Theta^{[1]}(H_\mu) \equiv 0 \pmod{p}.$$

Proof. We have by a direct calculation

$$\begin{aligned} S &= \begin{pmatrix} L & \frac{\mu}{2} \\ \frac{t\mu}{2} & M_0 \end{pmatrix} \\ &= \begin{pmatrix} 1_{n-j} & \frac{\mu}{2}M_0^{-1} \\ 0 & 1_j \end{pmatrix} \begin{pmatrix} L - \frac{1}{4}M_0^{-1}[t\mu] & 0 \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} 1_{n-j} & 0 \\ M_0^{-1}\frac{t\mu}{2} & 1_j \end{pmatrix}. \end{aligned}$$

(Multi-) linear algebra shows that for $J := \{n-j+1, \dots, n\}$ and $i, i' \in \{1, \dots, n-j\}$, we have

$$S_{(\{i\} \cup J, \{i'\} \cup J)}^{[j]} = \left(L - \frac{1}{4}M_0^{-1}[t\mu] \right)_{(i, i')}^{[1]} \cdot \det M_0.$$

Here the notation $S_{(P, Q)}^{[t]}$ is the same as in the proof of Theorem 3.3.

We observe that the left hand side is divisible by p and $\det M_0$ is coprime to p , therefore all entries of $L - \frac{1}{4}M_0^{-1}[t\mu]$ are divisible by p . \square

We return to the proof of Theorem 3.5. Let j be odd. By $\Theta^{[1]}(H_\mu) \equiv 0 \pmod{p}$, we can easily prove $\Theta^{[1]}(H_\mu^2) \equiv 0 \pmod{p}$. Then H_μ^2 is of weight $2k - j < 2p - 1$ and $2k - j$ is odd. Therefore $2k - j \not\equiv 0 \pmod{p-1}$. This implies $H_\mu^2 \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$ (see [8]). Hence we can apply Theorem 3.1 (1) to H_μ^2 . We conclude that $2k - j = p$ in the case where j is odd.

Let j be even and $k - j/2$ odd. The weight of H_μ is $k - j/2 < 2p - 1$ and therefore $H_\mu \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$, because of $2k - j \not\equiv 0 \pmod{p-1}$. In this case, we can directly apply Theorem 3.1 (1) to H_μ . Hence we obtain $k - j/2 = p$.

Let j be even and $k - j/2$ even. Note that $k - j/2 < 3p - 1$ by the assumption. If $H_\mu \equiv c \pmod{p}$ for some $c \in \mathbb{Z}_{(p)}$, then we have $k - j/2 \equiv 0 \pmod{p-1}$, otherwise we can apply Theorem 3.1 (1) to H_μ . Therefore, we obtain $k - j/2 \equiv 0 \pmod{p-1}$ or $k - j/2 = 2p$. This completes the proof of (1) in Theorem 3.5.

(2) Let $H_\mu \in M_{k-\frac{j}{2}}(\Gamma_1^{(n-j)}(4^2 N^2 Q^2))_{\mathbb{Z}_{(p)}}$ be the function appeared in the proof of (1). Note that $\omega_{4^2 N^2 Q^2}(H_\mu) + j/2 \leq \omega_N(F) \leq k$. By the assumption, we have

$$\begin{cases} \omega_{4^2 N^2 Q^2}(H_\mu^2) \leq 2k - j < p^2 - p + 1 & (j \text{ odd}), \\ \omega_{4^2 N^2 Q^2}(H_\mu) \leq k - j/2 < p^2 - p & (j \text{ even}) \end{cases} \quad \text{and} \quad p \mid (2k - j).$$

To apply Theorem 3.1 (2) to H_μ^2 (j odd) and H_μ (j even), we need to confirm that they are not constant modulo p .

If j is odd, then $2k - j \not\equiv 0 \pmod{p-1}$. This implies $H_\mu^2 \not\equiv c$ for any $c \in \mathbb{Z}_{(p)}$. Hence we can apply Theorem 3.1 (2) to H_μ^2 . It follows that

$$2k - j = \omega_{4^2 N^2 Q^2}(H_\mu^2) \leq 2\omega_{4^2 N^2 Q^2}(H_\mu) \leq 2\omega_N(F) - j \leq 2k - j.$$

This indicates $k = \omega_N(F)$.

Let j be even. Assume that $H_\mu \equiv c \pmod{p}$ for some $c \in \mathbb{Z}_{(p)}$. Then we have both conditions $k - j/2 \equiv 0 \pmod{p-1}$ and $p \mid (k - j/2)$ because of the assumption of the theorem. Then there exists $t \geq 1$ such that $2k - j = tp(p-1)$. However this is impossible because of $2k - j < p^2 - p$. This means that $H_\mu \not\equiv c \pmod{p}$ for any $c \in \mathbb{Z}_{(p)}$.

Hence we can apply Theorem 3.1 (2) to H_μ and then

$$k = \omega_{4^2 N^2 Q^2}(H_\mu) + j/2 \leq \omega_{4^2 N^2 Q^2}(F) \leq k.$$

Therefore we obtain $k = \omega_N(F)$.

This completes the proof of (2) in Theorem 3.5. \square

4 On operators $A^{(j)}(p)$

Following Choi-Choie-Richter [11], for a Siegel modular form $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ with a Fourier expansion $F = \sum_T a_F(T)q^T$, we define an operator $A(p)$ (their notation is $U(p)$) as

$$F|A(p) := \sum_{\substack{T \in \Lambda_n \\ p \mid \det T}} a_F(T)q^T.$$

We remark that this operator $A(p)$ is different from the usual $U(p)$ -type Hecke operator investigated in [4] and elsewhere. If $p \geq n + 3$, it is easy to see that $\widetilde{F|A(p)} = \widetilde{FE_{p-1}^{p+1}} - \Theta^{p-1}(\widetilde{F}) \in \widetilde{M}_{k+p^2-1}(\Gamma_n)_p$, because $\Theta(\widetilde{F}) \in \widetilde{M}_{k+p+1}(\Gamma_n)_p$ (see Theorem 2.7). Here $E_{p-1} \in M_{p-1}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ is such that $E_{p-1} \equiv 1 \pmod{p}$ obtained in [7]. Essentially, this formula appeared in Dewar-Richter [12]. It is not explicitly proved that $F|A(p)$ is a true modular form, we prove it here for more general operators $A^{(j)}(M)$.

Let M be a positive integer. For a formal Fourier series of the form $F = \sum_{T \in \Lambda_n} a_F(T) q^T$, we set

$$F|A^{(j)}(M) := \sum_{T^{[j]} \equiv 0 \pmod{M}} a_F(T) q^T.$$

Then we can prove its modularity as follows:

Theorem 4.1. Let k, j, n ($j \leq n$), M and N be positive integers. If $F \in M_k(\Gamma_1^{(n)}(N))$ then $F|A^{(j)}(N) \in M_k(\Gamma_1^{(n)}(NM^2))$. In particular, if $F \in M_k(\Gamma_0^{(n)}(N), \chi)$ for a Dirichlet character χ modulo N , then $F|A^{(j)}(M) \in M_k(\Gamma_0^{(n)}(NM^2), \chi)$.

Remark 4.2. As a special case in the above, we have

$$\begin{aligned} F|A^{(n)}(M) &= F|A(M) = \sum_{\substack{T \in \Lambda_n \\ M | \det T}} a_F(T) q^T, \\ F|A^{(1)}(M) &= F|U(M)V(M) = \sum_{T \in \Lambda_n} a_F(MT) q^{MT}. \end{aligned}$$

Here $U(M)$ and $V(M)$ are the usual operator described as

$$F|U(M) = \sum_{T \in \Lambda_n} a_F(MT) q^T, \quad F|V(M) = \sum_{T \in \Lambda_n} a_F(T) q^{MT}.$$

For more details, see [4].

Proof. We put $J := \{T \bmod N \mid T \in \Lambda_n\}$. Note that J is a finite set. Then we can find as in [8] that

$$\sum_{T \equiv T_0 \pmod{M}} a_F(T) q^T \in M_k(\Gamma_1^{(n)}(NM^2))$$

for any $T_0 \in J$. Now we consider

$$J_0^{(j)} := \{T \bmod M \mid T \in \Lambda_n, T^{[j]} \equiv 0 \pmod{M}\} \subset J.$$

Then we have

$$F|A^{(j)}(M) = \sum_{T_0 \in J_0^{(j)}} \sum_{T \equiv T_0 \pmod{M}} a_F(T) e^{2\pi i \text{tr}(TZ)}.$$

Hence $F|A^{(j)}(M) \in M_k(\Gamma_1^{(n)}(NM^2))$.

Assume that $F \in M_k(\Gamma_0^{(n)}(N), \chi)$. Using the standard procedure of twisting, we show that $F|A^{(j)}(M) \in M_k(\Gamma_0^{(n)}(NM^2), \chi)$. If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(NM^2)$ we get, using [8]

$$F|A^{(j)}(M)|_k g = \sum_{T_0 \in J_0^{(j)}} \sum_S F|_k \begin{pmatrix} 1_n & \frac{S}{M} \\ 0_n & 1_n \end{pmatrix} e^{-2\pi i \text{tr}(T_0 \frac{S}{M})}|_k g, \quad (4.1)$$

where S runs over all symmetric integral matrices of size n modulo M . Then we easily get

$$\begin{pmatrix} 1_n & \frac{S}{M} \\ 0_n & 1_n \end{pmatrix} \cdot g = \tilde{g} \cdot \begin{pmatrix} 1_n & \frac{\tilde{S}}{M} \\ 0_n & 1_n \end{pmatrix}$$

with $\tilde{g} \in \Gamma_0^{(n)}(N)$ and some integral symmetric \tilde{S} satisfying $\tilde{S} \equiv {}^t D S D \pmod{M}$. Note here that \tilde{g} satisfies $\tilde{g} \equiv \begin{pmatrix} * & * \\ * & D \end{pmatrix} \pmod{M}$. Keeping in mind that $A {}^t D \equiv 1_n \pmod{M}$ we may rewrite (4.1) as

$$\sum_{T_0} \sum_{\tilde{S}} (F|_k \tilde{g})|_k \begin{pmatrix} 1_n & \frac{\tilde{S}}{M} \\ 0_n & 1_n \end{pmatrix} e^{-2\pi i \text{tr}(-{}^t A T_0 A \frac{\tilde{S}}{M})}.$$

We observe that $F|_k \tilde{g} = \chi(\det D)F$ and $T_0 \mapsto \tilde{T}_0 := -{}^t A T_0 A$ just permutes the set $J_0^{(j)}$; this proves the assertion. \square

Remark 4.3. The proof actually shows that $F|A^{(j)}(M)$ is a modular form of level $\text{lcm}(M^2, N)$.

(2) We have the same statement as in Theorem 4.1 for F of half integral weight in the following way: We consider

$$G(Z) := F \cdot \theta^{(n)}(MZ),$$

where $\theta^{(n)}(Z)$ is the theta function introduced in Subsection 2.1. This is of integral weight and (obviously) we have

$$F|A^{(j)}(M) = (F|A^{(j)}(M)) \cdot \theta^{(n)}(MZ).$$

Therefore, the statement for F follows from that for G .

For any $F \in M_k(\Gamma_n)$, we have $F|A^{(j)}(p^m) \in M_k(\Gamma_0^{(n)}(\widetilde{p^{2m}}))$ by Theorem 4.1. By Theorem 2.3, we can regard as $A^{(j)}(p^m) : \widetilde{M}(\Gamma_n)_{p^l} \longrightarrow \widetilde{M}(\Gamma_n)_{p^l}$.

Proposition 4.4. For any $l \geq 1$, $m \geq 1$ and j ($1 \leq j \leq n$), we can decompose $\widetilde{M}(\Gamma_n)_{p^l}$ as

$$\widetilde{M}(\Gamma_n)_{p^l} = \text{Ker} A^{(j)}(p^m) \bigoplus \text{Im} A^{(j)}(p^m).$$

Proof. Let $\widetilde{F} \in \widetilde{M}(\Gamma_n)_{p^l}$. We set

$$\widetilde{F}_1 := \sum_{T^{[j]} \not\equiv 0 \pmod{p^m}} \widetilde{a_F(T)} q^T, \quad \widetilde{F}_2 := \sum_{T^{[j]} \equiv 0 \pmod{p^m}} \widetilde{a_F(T)} q^T.$$

Namely $\widetilde{F}_1 := \widetilde{F} - \widetilde{F}|A^{(j)}(p^m)$ and $\widetilde{F}_2 := \widetilde{F}|A^{(j)}(p^m)$. Then \widetilde{F} can be written as $\widetilde{F} = \widetilde{F}_1 + \widetilde{F}_2$. Then $\widetilde{F}_1 \in \text{Ker} A^{(j)}(p^m)$, $\widetilde{F}_2 \in \text{Im} A^{(j)}(p^m)$. This shows $\widetilde{M}(\Gamma_n)_{p^l} \subset \text{Ker} A^{(j)}(p^m) + \text{Im} A^{(j)}(p^m)$. The converse inclusion is trivial. Therefore

$$\widetilde{M}(\Gamma_n)_{p^l} = \text{Ker} A^{(j)}(p^m) + \text{Im} A^{(j)}(p^m).$$

We shall prove that the summation of the right hand side is direct. Let $\tilde{F} \in \text{Ker}A^{(j)}(p^m) \cap \text{Im}A^{(j)}(p^m)$. Then $\tilde{F} = \tilde{G}|A^{(j)}(p^m)$ for some $\tilde{G} \in \widetilde{M}(\Gamma_n)_{p^l}$. This implies

$$\tilde{F} = \sum_{T^{[j]} \equiv 0 \pmod{p^m}} \widetilde{a_G(T)} q^T.$$

On the other hand, it follows from $\tilde{F} \in \text{Ker}A^{(j)}(p^m)$ that

$$\tilde{F}|A^{(j)}(p^m) = \tilde{F} = 0.$$

Hence we have $\text{Ker}A^{(j)}(p^m) \cap \text{Im}A^{(j)}(p^m) = 0$. □

Remark 4.5. Similarly we have also

$$\widetilde{M}(\Gamma_n)_{p^l} = \text{Ker}\Theta^m \bigoplus \text{Im}\Theta^m,$$

for any l and m with $1 \leq l \leq m$.

We consider the action of $A^{(j)}(p^m)$ on the space of p -adic modular forms:

Proposition 4.6. If F is a p -adic modular form of degree n , then $F|A^{(j)}(p^m)$ is a p -adic modular form of degree n for any $1 \leq j \leq n$ and $m \geq 1$.

Proof. Since F is a p -adic modular form, there exists a sequence $\{G_l \in M_{k_l}(\Gamma_n)_{\mathbb{Z}_{(p)}}\}_l$ such that $F \equiv G_l \pmod{p^l}$. Then we have $F|A^{(j)}(p^m) \equiv G_l|A^{(j)}(p^m) \pmod{p^l}$. By Theorem 4.1, $G_l|A^{(j)}(p^m) \in M_{k_l}(\Gamma_0^{(n)}(p^{2m}))_{\mathbb{Z}_{(p)}}$ holds. By Theorem 2.3, $G_l|A^{(j)}(p^m)$ is also a p -adic modular form. Therefore $F|A^{(j)}(p^m)$ is a limit of a sequence of p -adic modular forms. This implies the assertion. □

5 Examples

In this section, we introduce some examples of elements of the mod p kernel of $\Theta^{[j]}$ and analyze the filtrations of some of them.

5.1 By the Siegel-Eisenstein series

Let $E_k^{(n)}$ be the Siegel-Eisenstein series of weight k of degree n , where $k > n + 1$ is an even integer. Let p be a prime and n a positive even integer such that $p \equiv (-1)^{\frac{n}{2}} \pmod{4}$ and $p > n + 3$. We set $k_{(n,p)} := \frac{n+p-1}{2}$. By Nagaoka's result [22], $E_{k_{(n,p)}}^{(n)}$ is an element of the mod p kernel of Θ . Note that this is mod p non-singular by the result of [5]. It follows from $k_{(n,p)} < p$ that

$$\omega_1 \left(E_{k_{(n,p)}}^{(n)} \right) = k_{(n,p)}.$$

As an easy application of Theorem 3.5 (1), we can prove $\Theta^{[n-1]} \left(E_{k(n,p)}^{(n)} \right) \not\equiv 0 \pmod{p}$ as follows: We can find an integer $1 \leq j_0 \leq n-1$ such that j_0 is the max of positive integers j satisfying $\Theta^{[j]} \left(E_{k(n,p)}^{(n)} \right) \not\equiv 0 \pmod{p}$. Applying Theorem 3.5 (1), we have $n + p - 1 - j_0 = p$. This implies $j_0 = n - 1$.

5.2 By theta series

In [6], we use certain theta series attached to quadratic forms to construct several types of modular forms in the kernel of theta operators mod p . We compute ω_N for some cases, here N can be an arbitrary number coprime to p . In this way we confirm that the constructions in [6] are the best possible ones in the sense that the level one forms obtained are of smallest possible weight.

First case: Here S is an even positive definite quadratic form of (even) rank n , exact level p and $\det(S) = p^2$.

We showed that the normalized theta series

$$\theta_S^{(n)}(Z) := \frac{1}{\sharp \text{Aut}_{\mathbb{Z}}(S)} \sum_{X \in \mathbb{Z}^{(n,n)}} e^{\pi i \text{tr}({}^t X S X Z)}$$

is congruent mod p to a level one form F of weight

$$k = \frac{n}{2} + (p-1),$$

where

$$\text{Aut}_{\mathbb{Z}}(S) = \{A \in \mathbb{Z}^{(n,n)} \mid {}^t A S A = S\}.$$

Then $a_F(S) = 1$, in particular, $F \not\equiv 0 \pmod{p}$ and

$$\Theta^{[n-1]}(F) \equiv 0 \pmod{p}, \quad \Theta^{[n-2]}(F) \not\equiv 0 \pmod{p}.$$

Here the second statement follows from

$$S^{[n-2]} \cdot a_F(S) \not\equiv 0 \pmod{p}.$$

Since $j = n - 2$ (even) and

$$k - \frac{j}{2} = k - \frac{n-2}{2} = p < p^2 - p,$$

we can apply Theorem 3.1 (2) to F . This implies

$$k = \omega_1(F).$$

Second case: Here S is an even positive quadratic form of (even) rank n , exact level p with $\det(S) = p$. In this case, $\theta_S^{(n)}$ is congruent mod p to a level one form of weight

$$k = \frac{n}{2} + \frac{p-1}{2}.$$

Then $\Theta^{[n]}(F) \equiv 0 \pmod{p}$ but $\Theta^{[n-1]}(F) \not\equiv 0 \pmod{p}$. Since $j = n - 1$ (odd) and $2k - j = 2k - n + 1 = p < p^2 - p + 1$, we can apply Theorem 3.1 (2). Therefore, in this case also we have

$$k = \omega_1(F).$$

Second case, with harmonic polynomial: Let S be as before and consider

$$\theta_{S,\det}^{(n)}(Z) := \sum_{X \in \mathbb{Z}^{(n,n)}} \det(X) e^{\pi i \text{tr}(^t X S X Z)}$$

Here we must assume in addition that $\text{Aut}_{\mathbb{Z}}(S)$ does not contain improper automorphisms (i.e. all automorphisms have determinant $+1$). Then it was shown in [6] that this theta series is congruent to a (cuspidal) level one modular form F of weight

$$k = \frac{n}{2} + 1 + 3\frac{p-1}{2}.$$

The proof was much more complicated than in the other cases. Again F satisfies $\Theta^{[n]}(F) \equiv 0 \pmod{p}$, but $\Theta^{[n-1]}(F) \not\equiv 0 \pmod{p}$.

In this case, from $j = n - 1$ (odd) we have

$$2k - j = n + 2 + 3(p - 1) - (n - 1) = 3p.$$

Then $p \mid (2k - j)$ and $2k - j = 3p < p^2 - p + 1$ (when $p \geq 5$). Applying Theorem 3.5 (2), we have

$$k = \omega_1(F).$$

Remark 5.1. There is a missing case here, namely $\theta_{S,\det}^{(n)}$ with S of level p and $\det(S) = p^2$. Here we do not know yet a good explicit construction of a level one form F congruent to $\theta_{S,\det}^{(n)}$ with low weight. A consideration similar to the one above suggests that

$$\omega_1(F) = \frac{n}{2} + 1 + 2(p - 1)$$

should hold.

5.3 By the operators $A^{(j)}(p)$

For any $F \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$, we have $\Theta^{[j]}(F|A^{(j)}(p)) \equiv 0 \pmod{p}$ by the definition of $A^{(j)}(p)$. Namely, we can always construct elements of the mod p kernel of $\Theta^{[j]}$ for any prime p . In particular, if $p \geq n + 3$ and $j = n$, then we get of weight $k + p^2 - 1$

for any $k \in \mathbb{Z}_{\geq 1}$ (see Section 4). Moreover these examples are not necessarily of type (b) introduced in Subsection 3.1. Because, we take a suitable F , then there exists $T \in \Lambda_n$ with $p \nmid \varepsilon(T)$ such that $a_{F|A^{(j)}(p)}(T) \not\equiv 0 \pmod{p}$. We remark that F is an element of the mod p kernel of $\Theta^{[j]}$ if and only if $F|A^{(j)}(p) \equiv F \pmod{p}$.

Let $X_{10}^{(2)}, X_{12}^{(2)}$ be cusp forms of degree 2, level 1 and weight 10, 12 respectively. We normalize them so that $a_{X_{10}} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = a_{X_{12}} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} = 1$.

Example 5.2. We have by direct calculation

$$\omega_1(E_8^{(2)}|A(7)) = 32, \quad \omega_1(E_{10}^{(2)}|A(7)) = 4, \quad \omega_1(X_{10}|A(7)) = 46.$$

All of these formulas satisfy $p \mid (2\omega_1(F) - n + 1)$. Therefore Conjecture 3.6 is true for these examples.

We introduce more examples of $\omega_1(F|A(p))$ in tables of Section 6. We shall explain the tables: Let $p \geq 5$ be a prime number and $R_p = \mathbb{F}_p[x_4, x_6, x_{10}, x_{12}]$ a polynomial ring over \mathbb{F}_p . For a positive integer k , we denote by $R_{p,k} \subset R_p$ the space of isobaric polynomials of weight k . We define a linear map $\psi_k : R_{p,k} \rightarrow \widetilde{M}_k(\Gamma_2)_p$ by $\psi_k(f(x_4, x_6, x_{10}, x_{12})) = f(\widetilde{E}_4^{(2)}, \widetilde{E}_6^{(2)}, \widetilde{X}_{10}, \widetilde{X}_{12})$. Then by Nagaoka [23], ψ_k is an isomorphism. Let $H \in \widetilde{M}_{p-1}(\Gamma_2)_p$ be a modular form with $H = 1$. Therefore

$$H = \begin{cases} \widetilde{E}_4^{(2)} & \text{if } p = 5, \\ \widetilde{E}_6^{(2)} & \text{if } p = 7. \end{cases}$$

Note that $\psi_k^{-1}(H)$ is a prime element of R_p . For $F \in \widetilde{M}_k(\Gamma_2)_p$, denote by $\text{ord}_H(F)$ the maximum integer e such that $\psi_k^{-1}(F)/(\psi_k^{-1}(H))^e \in R_p$. We understand $\text{ord}_H(0) = \infty$.

We compute images of the linear operator

$$A(p) : \widetilde{M}_k(\Gamma_2)_p \rightarrow \widetilde{M}_{k+p^2-1}(\Gamma_2)_p$$

for a basis of $\widetilde{M}_k(\Gamma_2)_p$ for even $k \leq 60$ and $p = 5, 7$. We fix a basis $\mathcal{B}_{k,p} = \{F_1, \dots, F_m\}$ of $\widetilde{M}_k(\Gamma_2)_p$ so that

$$\text{ord}_H \left(\sum_{G \in S} a_G G \right) = \min \{ \text{ord}_H(a_G G) \mid G \in S \}, \quad (5.1)$$

for any choice of $a_G \in \mathbb{F}_p$ for each $G \in S$. Here $S = \{F|A(p) : F \in \mathcal{B}_{k,p}\}$. We can take such a basis as follows. In general, let R be a polynomial ring over a field and h a non-zero element of R . We fix a monomial order of R . Let \mathcal{M} be a subspace of R over K spanned by monomials which are not divisible by the initial term of h . Since $\{h\}$ is a Gröbner basis of the ideal Rh , we can perform the reduction algorithm uniquely. That is, for any $f \in R$, we can uniquely write f as

$$f = \sum_{i=0}^{\infty} g_i(f) h^i, \quad (5.2)$$

where $g_i(f) \in \mathcal{M}$ and $g_i(f) = 0$ for sufficiently large i . Take a basis $\mathcal{B} = \{F_1, \dots, F_m\}$ of $\widetilde{M}_k(\Gamma_2)_p$. We put $f_i = \psi_{k+p^2-1}^{-1}(F_i|A(p))$ and denote $g_j(f_i)$ by the element of \mathcal{M} as in (5.2) for $h = \psi_{p-1}^{-1}(H)$. Take a positive integer a so that $g_j(f_i) = 0$ for all $j > a$. Let \mathcal{M}' be a subspace of \mathcal{M} spanned by $\{g_j(f_i)\}_{1 \leq i \leq m, 0 \leq j \leq a}$. We fix a linear isomorphism $\Psi : \mathcal{M}' \cong \mathbb{F}_p^\nu$ and put $v(f_i) = \Psi(g_0(f_i)) \oplus \Psi(g_1(f_i)) \oplus \dots \oplus \Psi(g_a(f_i)) \in \mathbb{F}_p^{\nu(a+1)}$. If we take a basis \mathcal{B} so that the matrix $(v(f_1), \dots, v(f_m))$ is an echelon form, then the basis satisfies the condition (5.1).

For example, we take $\mathcal{B}_{18,5}$ as

$$\mathcal{B}_{18,5} = \left\{ \widetilde{E}_4^3 \widetilde{E}_6, \widetilde{E}_4^3 \widetilde{E}_6 + 2\widetilde{E}_6^3, \widetilde{E}_6 \widetilde{X}_{12}, \widetilde{E}_4^2 \widetilde{X}_{10} \right\}.$$

Here we simply write $E_k^{(2)}$ as E_k . Then its images are given as

$$\begin{aligned} \widetilde{E}_4^3 \widetilde{E}_6 | A(5) &= \widetilde{E}_4^6 \widetilde{E}_6 (\widetilde{E}_4^3 - 2\widetilde{X}_{12}) \\ (\widetilde{E}_4^3 \widetilde{E}_6 + 2\widetilde{E}_6^3) | A(5) &= \widetilde{E}_4^6 (\widetilde{E}_4^3 \widetilde{E}_6 + 2\widetilde{E}_6^3 + \widetilde{E}_4^2 \widetilde{X}_{10}), \\ \widetilde{E}_6 \widetilde{X}_{12} | A(5) &= \widetilde{E}_4^2 \widetilde{X}_{10} | A(5) = 0. \end{aligned}$$

We omit the explicit description of $\mathcal{B}_{k,p}$ for other cases. We note that the multiset

$$\{\text{ord}_H(F|A(p)) \mid F \in \mathcal{B}_{k,p}\}$$

does not depend on the choice of $\mathcal{B}_{k,p}$ satisfying (5.1). Define a map $\alpha_{k,p} : \mathcal{B}_{k,p} \rightarrow \mathbb{Z}^2 \times \mathbb{F}_p^2$ by

$$\alpha_{k,p}(F) = (\text{ord}_H(F|A(p)), l, l \bmod p, 2l - 1 \bmod p),$$

where $l = \omega_1(F|A(p))$. Tables 1 and 2 show the multiset

$$\{\alpha_{k,p}(F) \mid F \in \mathcal{B}_{k,p}\}.$$

Each element $[(a, b, c, d), e]$ in tables means that there exists exactly e modular forms $F \in \mathcal{B}_{18,5}$ such that $\alpha_{k,p}(F) = (a, b, c, d)$.

Examples show that there exists a modular form F with $\omega_1(F|A(p)) \not\equiv 0 \bmod p$ and $2\omega_1(F|A(p)) - 1 \not\equiv 0 \bmod p$. Filtrations of such modular forms in these tables are $\{24, 42, 54\}$ if $p = 5$ and $\{24, 48, 52\}$ if $p = 7$. For example, a modular form of degree 2, weight 24, level 1

$$F = E_4^3 E_6^2 + 2E_6^4 + 3E_4^2 E_6 X_{10} + 3E_4 X_{10}^2 + 2E_6^2 X_{12} + 3X_{12}^2$$

satisfies $\Theta^{[2]}(F) \equiv 0 \bmod 5$ and $\Theta^{[1]}(F) \not\equiv 0 \bmod 5$, but we have

$$2\omega_1(F) - 1 \not\equiv 0 \bmod 5.$$

Bold elements in tables indicate those modular forms.

5.4 More examples of filtrations

In this subsection, we show example of $\omega_1(F)$ for $F \in M_k(\Gamma_2)_{\mathbb{Z}(p)}$ with $\Theta(F) \equiv 0 \pmod p$ to test the validity of Conjecture 3.6. Here we compute the kernel of $\Theta : \widetilde{M}_k(\Gamma_2)_p \rightarrow \widetilde{M}_{k+p+1}(\Gamma_2)_p$ for $p < 80$ and an even $k \leq 100$ with $b_{k+p+1} \leq 15$. Here

$$b_k = \begin{cases} [k/10] & \text{if } k \text{ is even,} \\ [(k-5)/10] & \text{if } k \text{ is odd.} \end{cases}$$

Note that b_k gives the Sturm bound for $\widetilde{M}_k(\Gamma_2)_p$ (cf. [10, 18]). We take a basis $\mathcal{B} = \{F_1, \dots, F_m\}$ of $\text{Ker}\Theta$ so that the condition (5.1) holds for $S = \mathcal{B}$.

We understand $\text{ord}_H(F)$ for $F \in \widetilde{M}_k(\Gamma_2)_p$ with an odd k as follows: By Nagaoka [24], there uniquely exists $G \in \widetilde{M}_{k-35}(\Gamma_2)_p$ such that $F = X_{35}G$, where X_{35} is the Igusa's generator of weight 35. Then we define $\text{ord}_H(F) = \text{ord}_H(G)$.

Then we have computed the filtration $\omega_1(F_i)$ for $i = 1, \dots, m$. The table 3 is of these filtrations. The meaning of the table is as follows: For a prime p , a positive integer k appears in the corresponding cell if and only if $k \leq 100$, $b_{k+p+1} \leq 15$ and there exists $F \in \widetilde{M}_k(\Gamma_2)_p$ such that $F \neq 0$, $\omega_1(F) = k$ and $\Theta(F) = 0$.

Table 3 also shows that there exists $F \in \widetilde{M}_k(\Gamma_2)_p$ such that $\Theta(F) = 0$, $\omega_1(F) \not\equiv 0 \pmod p$ and $2\omega_1(F) - 1 \not\equiv 0 \pmod p$. The pairs $(p, \omega_1(F))$ for such F in the table are

$$(5, 24), (5, 42), (5, 54), (5, 66), (5, 72), (5, 74), (5, 84), (5, 92), (5, 96), \\ (7, 24), (7, 48), (7, 52), (7, 72), (7, 76), (7, 80), (7, 94), (7, 96), (11, 60), (13, 84).$$

6 Tables

6.1 Tables for filtrations of images of $A(p)$

The following tables are of $[(a, b, c, d), e]$, where

$$(a, b, c, d) := \alpha_{k,p}(F) = (\text{ord}_H(F|A(p)), l, l \pmod p, 2l - 1 \pmod p),$$

$l = \omega_1(F|A(p))$, and e is the number of modular forms which have $\alpha_{k,p}(F)$. For more details, see Subsection 5.3

Table 1: Filtrations of images of $A(5)$

4	$[(7, 0, 0, 4), 1]$
6	$[(3, 18, 3, 0), 1]$
8	$[(8, 0, 0, 4), 1]$
10	$[(4, 18, 3, 0), 1], [(\infty, 0, 0, 4), 1]$
12	$[(2, 28, 3, 0), 1], [(9, 0, 0, 4), 1], [(\infty, 0, 0, 4), 1]$
14	$[(5, 18, 3, 0), 1], [(\infty, 0, 0, 4), 1]$

16	$[(3, 28, 3, 0), 1], [(10, 0, 0, 4), 1], [(\infty, 0, 0, 4), 2]$
18	$[(6, 18, 3, 0), 2], [(\infty, 0, 0, 4), 2]$
20	$[(4, 28, 3, 0), 2], [(11, 0, 0, 4), 1], [(\infty, 0, 0, 4), 2]$
22	$[(2, 38, 3, 0), 1], [(7, 18, 3, 0), 2], [(\infty, 0, 0, 4), 3]$
24	$[(0, 48, 3, 0), 2], [(5, 28, 3, 0), 2], [(\mathbf{6}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(12, 0, 0, 4), 1], [(\infty, 0, 0, 4), 2]$
26	$[(3, 38, 3, 0), 1], [(8, 18, 3, 0), 2], [(\infty, 0, 0, 4), 4]$
28	$[(1, 48, 3, 0), 2], [(6, 28, 3, 0), 2], [(\mathbf{7}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(13, 0, 0, 4), 1], [(\infty, 0, 0, 4), 4]$
30	$[(4, 38, 3, 0), 1], [(6, 30, 0, 4), 1], [(9, 18, 3, 0), 2], [(\infty, 0, 0, 4), 7]$
32	$[(2, 48, 3, 0), 3], [(7, 28, 3, 0), 2], [(\mathbf{8}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(14, 0, 0, 4), 1], [(\infty, 0, 0, 4), 5]$
34	$[(0, 58, 3, 0), 3], [(5, 38, 3, 0), 1], [(7, 30, 0, 4), 1], [(10, 18, 3, 0), 2], [(\infty, 0, 0, 4), 7]$
36	$[(3, 48, 3, 0), 4], [(8, 28, 3, 0), 2], [(\mathbf{9}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(15, 0, 0, 4), 1], [(\infty, 0, 0, 4), 9]$
38	$[(1, 58, 3, 0), 3], [(6, 38, 3, 0), 1], [(8, 30, 0, 4), 1], [(11, 18, 3, 0), 2], [(\infty, 0, 0, 4), 9]$
40	$[(4, 48, 3, 0), 5], [(9, 28, 3, 0), 2], [(\mathbf{10}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(16, 0, 0, 4), 1], [(\infty, 0, 0, 4), 12]$
42	$[(2, 58, 3, 0), 4], [(\mathbf{6}, \mathbf{42}, \mathbf{2}, \mathbf{3}), \mathbf{1}], [(7, 38, 3, 0), 1], [(9, 30, 0, 4), 1], [(12, 18, 3, 0), 2], [(\infty, 0, 0, 4), 13]$
44	$[(0, 68, 3, 0), 3], [(5, 48, 3, 0), 5], [(10, 28, 3, 0), 2], [(\mathbf{11}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(17, 0, 0, 4), 1], [(\infty, 0, 0, 4), 12]$
46	$[(3, 58, 3, 0), 5], [(\mathbf{7}, \mathbf{42}, \mathbf{2}, \mathbf{3}), \mathbf{1}], [(8, 38, 3, 0), 1], [(10, 30, 0, 4), 1], [(13, 18, 3, 0), 2], [(\infty, 0, 0, 4), 17]$
48	$[(1, 68, 3, 0), 3], [(6, 48, 3, 0), 7], [(11, 28, 3, 0), 2], [(\mathbf{12}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(18, 0, 0, 4), 1], [(\infty, 0, 0, 4), 17]$
50	$[(4, 58, 3, 0), 5], [(6, 50, 0, 4), 1], [(\mathbf{8}, \mathbf{42}, \mathbf{2}, \mathbf{3}), \mathbf{1}], [(9, 38, 3, 0), 1], [(11, 30, 0, 4), 1], [(14, 18, 3, 0), 2], [(\infty, 0, 0, 4), 20]$
52	$[(2, 68, 3, 0), 4], [(7, 48, 3, 0), 7], [(12, 28, 3, 0), 2], [(\mathbf{13}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(19, 0, 0, 4), 1], [(\infty, 0, 0, 4), 22]$
54	$[(0, 78, 3, 0), 6], [(5, 58, 3, 0), 5], [(\mathbf{6}, \mathbf{54}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(7, 50, 0, 4), 1], [(\mathbf{9}, \mathbf{42}, \mathbf{2}, \mathbf{3}), \mathbf{1}], [(10, 38, 3, 0), 1], [(12, 30, 0, 4), 1], [(15, 18, 3, 0), 2], [(\infty, 0, 0, 4), 21]$
56	$[(3, 68, 3, 0), 4], [(5, 60, 0, 4), 1], [(8, 48, 3, 0), 7], [(13, 28, 3, 0), 2], [(\mathbf{14}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(20, 0, 0, 4), 1], [(\infty, 0, 0, 4), 26]$
58	$[(1, 78, 3, 0), 7], [(6, 58, 3, 0), 5], [(\mathbf{7}, \mathbf{54}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(8, 50, 0, 4), 1], [(\mathbf{10}, \mathbf{42}, \mathbf{2}, \mathbf{3}), \mathbf{1}], [(11, 38, 3, 0), 1], [(13, 30, 0, 4), 1], [(16, 18, 3, 0), 2], [(\infty, 0, 0, 4), 27]$
60	$[(4, 68, 3, 0), 4], [(6, 60, 0, 4), 2], [(9, 48, 3, 0), 7], [(14, 28, 3, 0), 2], [(\mathbf{15}, \mathbf{24}, \mathbf{4}, \mathbf{2}), \mathbf{1}], [(21, 0, 0, 4), 1], [(\infty, 0, 0, 4), 35]$

Table 2: Filtrations of images of $A(7)$

4	$[(8, 4, 4, 0), 1]$
6	$[(9, 0, 0, 6), 1]$
8	$[(4, 32, 4, 0), 1]$
10	$[(2, 46, 4, 0), 1], [(9, 4, 4, 0), 1]$
12	$[(0, 60, 4, 0), 2], [(10, 0, 0, 6), 1]$
14	$[(5, 32, 4, 0), 1], [(\infty, 0, 0, 6), 1]$
16	$[(3, 46, 4, 0), 2], [(10, 4, 4, 0), 1], [(\infty, 0, 0, 6), 1]$
18	$[(1, 60, 4, 0), 2], [(11, 0, 0, 6), 1], [(\infty, 0, 0, 6), 1]$
20	$[(6, 32, 4, 0), 2], [(\infty, 0, 0, 6), 3]$
22	$[(4, 46, 4, 0), 2], [(11, 4, 4, 0), 1], [(\infty, 0, 0, 6), 3]$
24	$[(2, 60, 4, 0), 3], [(\mathbf{8}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(12, 0, 0, 6), 1], [(\infty, 0, 0, 6), 3]$
26	$[(0, 74, 4, 0), 2], [(7, 32, 4, 0), 2], [(\infty, 0, 0, 6), 3]$
28	$[(5, 46, 4, 0), 2], [(8, 28, 0, 6), 1], [(12, 4, 4, 0), 1], [(\infty, 0, 0, 6), 6]$
30	$[(3, 60, 4, 0), 4], [(\mathbf{9}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(13, 0, 0, 6), 1], [(\infty, 0, 0, 6), 5]$
32	$[(1, 74, 4, 0), 3], [(8, 32, 4, 0), 3], [(\infty, 0, 0, 6), 6]$
34	$[(6, 46, 4, 0), 3], [(9, 28, 0, 6), 1], [(13, 4, 4, 0), 1], [(\infty, 0, 0, 6), 9]$
36	$[(4, 60, 4, 0), 6], [(\mathbf{10}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(14, 0, 0, 6), 1], [(\infty, 0, 0, 6), 9]$
38	$[(2, 74, 4, 0), 4], [(9, 32, 4, 0), 3], [(\infty, 0, 0, 6), 9]$
40	$[(0, 88, 4, 0), 7], [(7, 46, 4, 0), 3], [(10, 28, 0, 6), 1], [(14, 4, 4, 0), 1], [(\infty, 0, 0, 6), 9]$
42	$[(5, 60, 4, 0), 7], [(\mathbf{11}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(15, 0, 0, 6), 1], [(\infty, 0, 0, 6), 13]$
44	$[(3, 74, 4, 0), 6], [(10, 32, 4, 0), 3], [(\infty, 0, 0, 6), 15]$
46	$[(1, 88, 4, 0), 8], [(8, 46, 4, 0), 4], [(11, 28, 0, 6), 1], [(15, 4, 4, 0), 1], [(\infty, 0, 0, 6), 13]$
48	$[(6, 60, 4, 0), 7], [(\mathbf{8}, \mathbf{48}, \mathbf{6}, \mathbf{4}), \mathbf{1}], [(\mathbf{12}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(16, 0, 0, 6), 1], [(\infty, 0, 0, 6), 21]$
50	$[(4, 74, 4, 0), 7], [(11, 32, 4, 0), 3], [(\infty, 0, 0, 6), 21]$
52	$[(2, 88, 4, 0), 9], [(\mathbf{8}, \mathbf{52}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(9, 46, 4, 0), 4], [(12, 28, 0, 6), 1], [(16, 4, 4, 0), 1], [(\infty, 0, 0, 6), 21]$
54	$[(0, 102, 4, 0), 8], [(7, 60, 4, 0), 7], [(\mathbf{9}, \mathbf{48}, \mathbf{6}, \mathbf{4}), \mathbf{1}], [(\mathbf{13}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(17, 0, 0, 6), 1], [(\infty, 0, 0, 6), 21]$
56	$[(5, 74, 4, 0), 8], [(8, 56, 0, 6), 1], [(12, 32, 4, 0), 3], [(\infty, 0, 0, 6), 30]$
58	$[(3, 88, 4, 0), 11], [(\mathbf{9}, \mathbf{52}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(10, 46, 4, 0), 4], [(13, 28, 0, 6), 1], [(17, 4, 4, 0), 1], [(\infty, 0, 0, 6), 28]$
60	$[(1, 102, 4, 0), 10], [(8, 60, 4, 0), 9], [(\mathbf{10}, \mathbf{48}, \mathbf{6}, \mathbf{4}), \mathbf{1}], [(\mathbf{14}, \mathbf{24}, \mathbf{3}, \mathbf{5}), \mathbf{1}], [(18, 0, 0, 6), 1], [(\infty, 0, 0, 6), 30]$

6.2 Table for filtrations of the kernel of $\Theta^{[2]}$

Table 3 shows filtration $\omega_1(F)$ for $F \in M_k(\Gamma_2)_{\mathbb{Z}(p)}$ with $\Theta^{[2]}(F) \equiv 0 \pmod{p}$, $k \leq 100$, $p < 80$ and $b_{k+p+1} \leq 15$. For more details, see Subsection 5.4.

Table 3: Filtrations of the kernel of $\Theta^{[2]}$

p	k
5	0, 18, 24, 28, 30, 38, 42, 48, 50, 54, 58, 60, 66, 68, 72, 74, 78, 80, 83, 84, 88, 90, 92, 93, 96, 98
7	0, 4, 24, 28, 32, 46, 48, 52, 56, 60, 70, 72, 74, 76, 80, 81, 84, 88, 94, 95, 96, 98
11	0, 6, 28, 44, 50, 60, 66, 72, 83, 88, 94
13	0, 20, 46, 52, 59, 72, 78, 84, 98
17	0, 26, 60, 68, 77, 94
19	0, 10, 48, 76, 86
23	0, 12, 35, 58, 92
29	0, 44
31	0, 16, 47, 78
37	0, 56, 93
41	0, 62
43	0, 22
47	0, 24, 71
53	0, 80
59	0, 30, 89
61	0, 92
67	0, 34
71	0, 36
73	0
79	0, 40

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